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Almost sure exponential stability sensitive to small time delay of stochastic neutral functional differential equations ☆

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ABSTRACT

In this work, we establish a theory about the almost sure pathwise exponential stability property for a class of stochastic neutral functional differential equations by developing a semigroup scheme for the drift part of the systems under consideration and dealing with their pathwise stability through a perturbation approach, rather than through that one to get their moment stability first. As an illustration, we can show that some stochastic systems have their almost sure exponential stability not sensitive to small delays.

1. Introduction

Let H be a real Hilbert space with its norm $\|\cdot\|_H$ and inner product $\langle\cdot,\cdot\rangle_H$, respectively. We denote by $\mathscr{L}(H)$ the space of all bounded linear operators from H into itself. Let $r>0$ be a constant and consider a deterministic time delay differential equation in H,

$$\begin{cases} dy(t) = \big[Ay(t) + By(t-r)\big]dt, & t\geq 0,\\ y(0)=\phi_0\in H,\ y(t)=\phi_1(t),\ t\in[-r,0], & \phi_1\in L^2([-r,0],H), \end{cases} \tag{1.1}$$

where A is a linear operator generating a C_0-semigroup e^{tA}, $t\geq 0$, on H and B is some appropriate linear operator in H. Recall that the trivial solution of (1.1) is called exponentially stable if there exist number $M=M(\phi)\geq 1$ and constant $\gamma>0$ such that $\|y(t)\|_H\leq M(\phi)e^{-\gamma t}$ for all $t\geq 0$. It was observed by Datko et al. [1] (see also [2]) that small delays may destroy exponential stability of an infinite dimensional system like (1.1). More precisely, if the spectrum $\sigma(A)$ of A is unbounded along an imaginary line, it was shown (see Theorem 7.4, [3]) that one can find a bounded linear operator $B\in\mathscr{L}(H)$ such that $A+B$ generates an exponentially stable semigroup, i.e., $\|e^{t(A+B)}\|\leq Ce^{-\beta t}$, C, $\beta>0$, for all $t\geq 0$, and meanwhile for any $\varepsilon>0$, there always exists $r\in(0,\varepsilon)$ such that the system (1.1) is not exponentially stable. From this

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observation, we can see that the unboundedness of the spectrum set of A along imaginary axes may cause trouble for exponential stability of (1.1). Thus, one need make additional assumptions on e^{tA}, $t \geq 0$, to obtain the desired stability. In fact, we have the following result whose proof is referred to [3].

Theorem 1.1. *Assume that A generates a norm continuous C_0-semigroup e^{tA}, $t \geq 0$, i.e., $e^{\cdot A} : [0,\infty) \to \mathscr{L}(H)$ is continuous and the semigroup generated by $A + B$, $B \in \mathscr{L}(H)$, is exponentially stable in H. Then there exists a constant $r_0 > 0$ such that the trivial solution of (1.1) is exponentially stable for all $r \in (0, r_0)$.*

Bearing Theorem 1.1 in mind, let $A = \partial^2/\partial\xi^2$ which surely generates a norm continuous C_0-semigroup on $H = L^2(0,\pi)$. Consider a linear partial differential equation in H,

$$\begin{cases} dy(t,\xi) = \dfrac{\partial^2}{\partial\xi^2} y(t,\xi)dt + \alpha y(t-r,\xi)dt, \quad t \geq 0, \quad \xi \in [0,\pi], \\ y(t,0) = y(t,\pi) = 0,\ t \geq 0,\ \ y(0,\cdot) = \phi_0(\cdot) \in L^2(0,\pi),\ \ y(\theta) = \phi_1(\theta) \in L^2([-r,0]; L^2(0,\pi)) \end{cases} \tag{1.2}$$

where $r \geq 0$ and $\alpha \in \mathbb{R}$. If $r = 0$, it is well known that when $\alpha < 1$, the trivial solution of (1.2) is exponentially stable. If $r \neq 0$ and $\alpha < 1$, we have by virtue of Theorem 1.1 that the trivial solution of (1.2) is exponentially stable when $r \in (0, r_0)$ for some $r_0 > 0$.

Next we turn our attention to stochastic systems and consider their sensitivity problem to small delays of almost sure pathwise exponential stability. As a motivation example, let us consider a stochastic version of (1.1) analogous to (1.2). Precisely, let $r \geq 0$ and consider a stochastic delay partial differential equation,

$$\begin{cases} dy(t,\xi) = \dfrac{\partial^2}{\partial\xi^2} y(t,\xi)dt + \alpha y(t-r,\xi)dt + \sigma y(t,\xi)dw(t), \quad t \geq 0, \quad \xi \in [0,\pi], \\ y(t,0) = y(t,\pi) = 0,\ t \geq 0,\ \ y(0) = \phi_0 \in L^2(0,\pi),\ y_0(\cdot) = \phi_1(\cdot) \in L^2([-r,0]; L^2(0,\pi)) \end{cases} \tag{1.3}$$

where α, $\sigma \in \mathbb{R}$ and w is a standard real Brownian motion. Let $H = L^2(0,\pi)$, then it is known (see, e.g., [4]) that when $r = 0$ and $\alpha - 1 < \sigma^2/2$, the trivial solution of (1.3) is exponentially stable in the almost sure sense, i.e., for each $\phi_0 \in H$, there exist constants $M \geq 1$, $\gamma > 0$ and a random variable $T = T(\phi_0,\omega) \geq 0$ such that $\|y(t,\cdot)\|_H \leq M\|\phi_0\|_H e^{-\gamma t}$ for all $t \geq T$ almost surely. If $r > 0$, it was shown in [4] that under the condition $\alpha - 1 < \sigma^2/2$, the pathwise exponential stability of the trivial solution to (1.3) is not sensitive to small delays $r > 0$. Precisely, in addition to the condition $\alpha - 1 < \sigma^2/2$, if it is further assumed that the delay parameter $r > 0$ satisfies $|\alpha| \exp(3\sigma^2 r/2) - 1 < \sigma^2/2$, or equivalently, satisfies

$$r < \frac{2}{3\sigma^2} \ln\Big(\frac{\frac{1}{2}\sigma^2 + 1}{|\alpha|}\Big), \tag{1.4}$$

when $\alpha \neq 0$, $\sigma \neq 0$, then the trivial solution of (1.3) is exponentially stable in almost sure sense. Now let us consider a time delay version of (1.3) of neutral type in the following form

$$\begin{cases} d(y(t,\xi) - \gamma y(t-r,\xi)) = \dfrac{\partial^2}{\partial\xi^2} y(t,\xi)dt + \alpha \dfrac{\partial^2}{\partial\xi^2} y(t-r,\xi)dt + \sigma y(t,\xi)dw(t), \quad t \geq 0, \quad \xi \in [0,\pi], \\ y(t,0) = y(t,\pi) = 0,\ t \geq 0,\ \ y(0) = \phi_0 \in L^2(0,\pi),\ y_0(\cdot) = \phi_1(\cdot) \in L^2([-r,0]; L^2(0,\pi)), \end{cases} \tag{1.5}$$

where $\gamma \in \mathbb{R}$. In comparison with (1.3), the novelty of system (1.5) is that a delay term is included under the differentiation at the left-hand side of (1.5) on one hand, and a time delay appears in the highest-order derivative term in (1.5) on the other. Here we want to know whether, in addition to some conditions on α, σ, γ, the trivial solution of Eq. (1.5) can still secure its pathwise exponential stability, at least for sufficiently small delay parameter $r > 0$. In this work, we shall consider this sensitivity problem of pathwise stability to small delays for such stochastic functional differential equations of neutral type as (1.5).

2. Strongly continuous semigroups

Let V be a real Hilbert space and $a : V \times V \to \mathbb{R}$ a bounded bilinear form satisfying the so-called Gårding's inequality $|a(x,y)| \le \beta\|x\|_V \cdot \|y\|_V$, $a(x,x) \le -\alpha\|x\|_V^2$ for all x, $y \in V$ and some constants $\beta > 0$, $\alpha > 0$. Let A be a linear operator associated with this form through $a(x,y) = \langle x, Ay\rangle_{V,V^*}$, x, $y \in V$, where V^* is the dual space of V and $\langle\cdot,\cdot\rangle_{V,V^*}$ is the dual pairing between V and V^*. Then $A \in \mathscr{L}(V, V^*)$, the family of all bounded and linear operators from V to V^*, and A generates a C_0-semigroup e^{tA}, $t \ge 0$, on V^*. We also introduce the standard interpolation Hilbert space $H = (V, V^*)_{1/2,2}$, which is described by $H = \{x \in V^* : \int_0^\infty \|Ae^{tA}x\|_{V^*}^2 dt < \infty\}$. We identify the dual H^* of H with H, then it is easy to see that $V \hookrightarrow H = H^* \hookrightarrow V^*$ where the embedding $\hookrightarrow$ is dense and continuous with $\|x\|_H^2 \le \nu\|x\|_V^2$, $x \in V$, for some constant $\nu \ge 1$. Hence, $\langle x, Ay\rangle_H = \langle x, Ay\rangle_{V,V^*}$ for all $x \in V$ and $y \in V$ with $Ay \in H$. It can be shown that the semigroup e^{tA}, $t \ge 0$, is bounded and analytic on both V^* and H such that $e^{tA} : V^* \to V$ for each $t > 0$ and for some constant $M_0 > 0$, $\|e^{tA}\|_{\mathscr{L}(V^*)} \le M_0$, and $\|e^{tA}\|_{\mathscr{L}(H)} \le e^{-\alpha t}$ for all $t \ge 0$.

Let $r > 0$ and $T \ge 0$. For $y \in L^2([-r,T], V)$, we always write $y_t(\theta) := y(t+\theta)$ for any $t \ge 0$ and $\theta \in [-r, 0]$ in this work. Now suppose that $D_1 \in \mathscr{L}(V)$, $D_2 \in \mathscr{L}(L^2([-r,0],V),V)$, $F_1 \in \mathscr{L}(V,V^*)$ and $F_2 \in \mathscr{L}(L^2([-r,0],V),V^*)$. We introduce two linear mappings D and F, respectively, defined by $Dy_t = D_1y(t-r) + D_2y_t$ and $Fy_t = F_1y(t-r) + F_2y_t$, $t \in [0,T]$, $y(\cdot) \in C([-r,T],V)$. It was shown in [5] that both D and F have a bounded linear extension on $L^2([-r,T],V)$, i.e., D, $F \in \mathscr{L}(L^2([-r,T],V), L^2([0,T],V))$. Let $\mathcal{H} = H \times L^2([-r,0],V)$, equipped with its canonical inner product $\langle\phi,\psi\rangle_{\mathcal{H}} = \langle\phi_0,\psi_0\rangle_H + \int_{-r}^0 \langle\phi_1(\theta),\psi_1(\theta)\rangle_V d\theta$, $\phi = (\phi_0,\phi_1)$, $\psi = (\psi_0,\psi_1) \in \mathcal{H}$. Consider the following functional integral equation of neutral type in V^*,

$$y(t) - Dy_t = e^{tA}\phi_0 + \int_0^t e^{(t-s)A}Fy_s ds,\ t \ge 0;\ \ y_0 = \phi_1,\ \phi = (\phi_0,\phi_1) \in \mathcal{H}. \tag{2.1}$$

We say that y is a *(strict) solution* of (2.1) in $[0,T]$ if $y \in L^2([0,T],V) \cap W^{1,2}([0,T],V^*)$ and Eq. (2.1) is satisfied almost everywhere in $[0,T]$, $T \ge 0$. Here $W^{1,2}([0,T],V^*)$ is the Sobolev space consisting of all functions $y : [0,T] \to V^*$ such that y and its first order distributional derivative are in $L^2([0,T],V^*)$. It can be shown (see [5]) that for arbitrarily given $\phi = (\phi_0,\phi_1) \in H \times L^2([-r,0],V)$ and $T \ge 0$, there exists a function $y(t) \in V$, $t \in [-r,T]$, which is the unique solution of Eq. (2.1) with $y_0 = \phi_1$ such that $y(\cdot) \in L^2([-r,T],V)$ and $y(\cdot) - Dy_\cdot \in L^2([0,T],V) \cap W^{1,2}([0,T],V^*) \subset C([0,T],H)$. Based on this solution, we further define a family of operators $\mathcal{S}(t) : \mathcal{H} \to \mathcal{H}$, $t \ge 0$, by $\mathcal{S}(t)\phi = (y(t) - Dy_t, y_t)$ for any $\phi \in \mathcal{H}$. Then it was shown in [5] that $t \to \mathcal{S}(t)$ is a C_0-semigroup on $\mathcal{H}$. Moreover, the generator $\mathcal{A}$ of $\mathcal{S}(t)$ or $e^{t\mathcal{A}}$, $t \ge 0$, is given by

$$\mathscr{D}(\mathcal{A}) = \{(\phi_0,\phi_1) \in \mathcal{H} : \phi_1 \in W^{1,2}([-r,0],V),\ \phi_0 = \phi_1(0) - D\phi_1 \in V,\ A\phi_0 + F\phi_1 \in H\} \tag{2.2}$$

and for each $\phi = (\phi_0,\phi_1) \in \mathscr{D}(\mathcal{A})$, $\mathcal{A}\phi = (A\phi_0 + F\phi_1, \phi_1') \in \mathcal{H}$.

Now let us consider the following deterministic functional differential equation of neutral type in V^*,

$$\begin{cases} y(t) - Gy(t-r) = e^{tA}\phi_0 + \displaystyle\int_0^t \int_{-r}^0 e^{(t-s)A} d\eta(\theta) y(s+\theta) d\theta ds, & t \ge 0, \\ y_0 = \phi_1,\ \phi = (\phi_0,\phi_1) \in \mathcal{H}, \end{cases} \tag{2.3}$$

where $G \in \mathscr{L}(V)$ and $\eta : [-r,0] \to \mathscr{L}(V,V^*)$ is of bounded variation.

Proposition 2.1. *Suppose that $\langle x, Ax\rangle_{V,V^*} \le -\alpha\|x\|_V^2$ for $x \in V$ and some $\alpha > 0$. Assume that $\|G\| + \|G\|^2 < 1$ and for some $\lambda \in (0,\alpha)$,*

$$\Big[\frac{|\eta|(0) - |\eta|(-r)}{(1-\|G\|)(1 - e^{2\lambda r}(\|G\| + \|G\|^2))} \int_{-r}^0 e^{-2\lambda\theta} d|\eta|(\theta)\Big]^{1/2} < \alpha - \lambda, \tag{2.4}$$

where $|\eta|(\theta)$ *is the total variation on* $[-r,\theta]$, $\theta \in [-r,0]$, *then there exists a constant* $M > 0$ *such that* $\|e^{t\mathcal{A}}\| \le Me^{-\frac{\lambda}{\nu}t}$ *for all* $t \ge 0$, *where* $\nu \ge 1$ *is the constant given in* $\|v\|_H \le \nu\|v\|_V$ *for any* $v \in V$.

Proof. Note that since $\|G\| + \|G\|^2 < 1$, it makes sense that $1 - \|G\| > 0$ and there exists a $\lambda > 0$ such that $1 - e^{2\lambda r}(\|G\| + \|G\|^2) > 0$. We show that there exists an equivalent inner product $(\cdot,\cdot)_{\mathcal{H}}$ on $\mathcal{H} = H \times L^2([-r,0];V)$ to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$ such that $(\mathcal{A}\phi,\phi)_{\mathcal{H}} \le -\lambda\nu^{-1}\|\phi\|_{\mathcal{H}}$ for all $\phi \in \mathscr{D}(\mathcal{A})$. Here $(\cdot,\cdot)_{\mathcal{H}}$ is defined by $(\phi,\psi)_{\mathcal{H}} := \langle\phi_0,\psi_0\rangle_H + \int_{-r}^0 \gamma(\theta)\langle\phi_1(\theta),\psi_1(\theta)\rangle_V d\theta$, $\phi,\ \psi \in \mathcal{H}$, where $\gamma : [-r,0] \to \mathbb{R}_+$ is given by

$$\gamma(\theta) = e^{2\lambda\theta}\Big[\alpha - \lambda - \frac{|\eta|(0) - |\eta|(-r)}{(1-\|G\|)(\alpha-\lambda)}\int_\theta^0 e^{-2\lambda\tau}d|\eta|(\tau)\Big], \qquad \theta \in [-r,0]. \tag{2.5}$$

First, note that under the conditions of Proposition 2.1, $(\cdot,\cdot)_{\mathcal{H}}$ does define an inner product. Indeed, both (2.4) and (2.5) imply the lower boundedness of $\gamma(\cdot)$,

$$\gamma(\theta) \ge e^{-2\lambda r}\Big[\alpha - \lambda - \frac{|\eta|(0) - |\eta(-r)|}{(1-\|G\|)(\alpha-\lambda)}\int_{-r}^0 e^{-2\lambda\tau}d|\eta|(\tau)\Big] \quad \text{for any} \quad \theta \in [-r,0].$$

This implies that $(\cdot,\cdot)_{\mathcal{H}}$ defines an inner product on $\mathcal{H}$. Also, it is easy to see that for any $\phi \in \mathcal{H}$,

$$(\phi,\phi)_{\mathcal{H}} = \|\phi_0\|_H^2 + \int_{-r}^0 \gamma(\theta)\|\phi_1(\theta)\|_V^2 d\theta \le [1 + e^{2\lambda r}(\alpha-\lambda)]\Big(\|\phi_0\|_H^2 + \int_{-r}^0 \|\phi_1(\theta)\|_V^2 d\theta\Big)$$

which implies that the inner product $(\cdot,\cdot)_{\mathcal{H}}$ is also equivalent to the canonical inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$. On the other hand, since $\langle x,y\rangle_H = \langle x,y\rangle_{V,V^*}$ for any $x \in V$, $y \in H$, it follows for any $\phi \in \mathscr{D}(\mathcal{A})$ that

$$\begin{aligned}
(\phi,(\mathcal{A}+\lambda\nu^{-1})\phi)_{\mathcal{H}} &= \Big\langle \phi_0, A\phi_0 + \lambda\nu^{-1}\phi_0 + \int_{-r}^0 d\eta(\theta)\phi_1(\theta)\Big\rangle_H + \int_{-r}^0 \gamma(\theta)\langle\phi_1(\theta),\dot\phi_1(\theta) + \lambda\nu^{-1}\phi_1(\theta)\rangle_V d\theta \\
&\le \Big\langle \phi_0, A\phi_0 + \int_{-r}^0 d\eta(\theta)\phi_1(\theta)\Big\rangle_{V,V^*} + \int_{-r}^0 \gamma(\theta)\langle\dot\phi_1(\theta),\phi_1(\theta)\rangle_V d\theta + \lambda\|\phi_0\|_V^2 + \lambda\int_{-r}^0 \gamma(\theta)\|\phi_1(\theta)\|_V^2 d\theta \qquad (2.6)\\
&\le (\lambda-\alpha)\|\phi_0\|_V^2 + \|\phi_0\|_V\int_{-r}^0 \|\phi_1(\theta)\|_V d|\eta|(\theta) + \int_{-r}^0 \gamma(\theta)\Big(\frac{1}{2}\frac{d}{d\theta}\|\phi_1(\theta)\|_V^2 + \lambda\|\phi_1(\theta)\|_V^2\Big)d\theta.
\end{aligned}$$

By using integration by parts, one can further derive from (2.6) and (2.4) that for $\phi \in \mathscr{D}(\mathcal{A})$,

$$\begin{aligned}
(\phi,(\mathcal{A}+\lambda\nu^{-1})\phi)_{\mathcal{H}} &\le (\lambda-\alpha)\|\phi_0\|_V^2 + \|\phi_0\|_V\int_{-r}^0 \|\phi_1(\theta)\|_V d|\eta|(\theta) + \frac{1}{2}(\alpha-\lambda)\|\phi_0 + G\phi_1(-r)\|_V^2 \\
&\quad - \frac{1}{2}\gamma(-r)\|\phi_1(-r)\|_V^2 - \frac{|\eta|(0) - |\eta|(-r)}{2(1-\|G\|)(\alpha-\lambda)}\int_{-r}^0 \|\phi_1(\theta)\|_V^2 d|\eta|(\theta) \\
&\le -\frac{1}{2}(\alpha-\lambda)(1-\|G\|)\|\phi_0\|_V^2 + \frac{1}{2}\Big(\|G\|(\alpha-\lambda) + (\alpha-\lambda)\|G\|^2 - \gamma(-r)\Big)\|\phi_1(-r)\|_V^2 \qquad (2.7)\\
&\quad + \|\phi_0\|_V\int_{-r}^0 \|\phi_1(\theta)\|_V d|\eta|(\theta) - \frac{|\eta|(0) - |\eta|(-r)}{2(1-\|G\|)(\alpha-\lambda)}\int_{-r}^0 \|\phi_1(\theta)\|_V^2 d|\eta|(\theta) \\
&\le -\frac{1}{2}(\alpha-\lambda)(1-\|G\|)\|\phi_0\|_V^2 + \|\phi_0\|_V\int_{-r}^0 \|\phi_1(\theta)\|_V d|\eta|(\theta) - \frac{|\eta|(0) - |\eta|(-r)}{2(1-\|G\|)(\alpha-\lambda)}\int_{-r}^0 \|\phi_1(\theta)\|_V^2 d|\eta|(\theta).
\end{aligned}$$

If $\|\phi_0\|_V = 0$ or $|\eta|(0) = 0$, i.e., η is constant, it is immediate from (2.7) that $(\phi,(\mathcal{A}+\lambda\nu^{-1})\phi)_{\mathcal{H}} \le 0$ for all $\phi \in \mathscr{D}(\mathcal{A})$. If $\|\phi_0\|_V \ne 0$ and $|\eta|(0) > 0$ for $\phi \in \mathscr{D}(\mathcal{A})$, we have from (2.7) that

$$\begin{aligned}
(\phi,(\mathcal{A}+\lambda\nu^{-1})\phi)_{\mathcal{H}} &\le \|\phi_0\|_V^2\int_{-r}^0 \Big[-\frac{(1-\|G\|)(\alpha-\lambda)}{2(|\eta|(0)-|\eta|(-r))} + \frac{\|\phi_1(\theta)\|_V}{\|\phi_0\|_V} - \frac{|\eta|(0)-|\eta|(-r)}{2(1-\|G\|)(\alpha-\lambda)}\frac{\|\phi_1(\theta)\|_V^2}{\|\phi_0\|_V^2}\Big]d|\eta|(\theta) \\
&= -\frac{\|\phi_0\|_V^2}{2}\cdot\frac{|\eta|(0)-|\eta|(-r)}{\alpha-\lambda}\int_{-r}^0 \Big(\frac{\|\phi_1(\theta)\|_V}{(1-\|G\|)^{1/2}\|\phi_0\|_V} - \frac{(1-\|G\|)^{1/2}(\alpha-\lambda)}{|\eta|(0)-|\eta|(-r)}\Big)^2 d|\eta|(\theta) \le 0.
\end{aligned}$$

Hence, it follows that $(\phi, (\mathcal{A} + \lambda\nu^{-1})\phi)_{\mathcal{H}} \leq 0$ for all $\phi \in \mathscr{D}(\mathcal{A})$. Due to the equivalence between $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $(\cdot,\cdot)_{\mathcal{H}}$, this further implies, in addition to Proposition 2.1.4 in [6], that C_0-semigroup $e^{t\mathcal{A}}$, $t \geq 0$, is exponentially stable. The proof is completed now. $\square$

3. Stochastic neutral functional evolution equations

Consider the following stochastic retarded evolution equation in the real Hilbert space H: for $t \geq 0$,

$$\begin{cases} y(t) - Gy(t-r) = e^{tA}\phi_0 + \displaystyle\int_0^t A_1 y(s-r)ds + \int_0^t \int_{-r}^0 A_2(\theta)y(s+\theta)d\theta ds + \int_0^t By(s)dw(s), \\ y_0 = \phi_1, \ \phi = (\phi_0, \phi_1) \in \mathcal{H}, \end{cases} \tag{3.1}$$

where $B \in \mathscr{L}(H)$, $G \in \mathscr{L}(V)$ with $\|G\| < 1$, $A_1 \in \mathscr{L}(V, V^*)$, $A_2(\cdot) \in L^2([-r,0]; \mathscr{L}(V, V^*))$ and $w(\cdot)$ is a standard real Brownian motion. Let $\mathcal{A}$ be the generator given in (2.2) and define a linear operator $\mathcal{B} \in \mathscr{L}(\mathcal{H})$ by $\mathcal{B}\phi = (B\phi_0, 0)$ for any $\phi \in \mathcal{H}$. Then Eq. (3.1) can be lifted up into a stochastic evolution equation without delays,

$$Y(t) = e^{t\mathcal{A}}\phi + \int_0^t e^{(t-s)\mathcal{A}}\mathcal{B}Y(s)dw(s), \quad t \geq 0; \ \ Y(0) = \phi, \quad \phi \in \mathcal{H}, \tag{3.2}$$

where $Y(t) = (y(t) - Gy(t-r), y_t)$, $t \geq 0$. It is easy to see that $\|\mathcal{B}\| = \|B\|$. We also want to employ the following result whose proof is referred to Proposition 2.1 in [4].

Proposition 3.1. *Suppose that there exist $\beta \in \mathbb{R}$, $\lambda > 0$ and $M \geq 1$ such that the C_0-semigroup $e^{t(\mathcal{A}+\beta\mathcal{B})}$ generated by $\mathcal{A} + \beta\mathcal{B}$ satisfies $\|e^{t(\mathcal{A}+\beta\mathcal{B})}\| \leq Me^{-\lambda t}$, $t \geq 0$ and $\beta^2 + 2M^2\|B\|^2 < 4\lambda$, then the solution of (3.2) is almost sure exponentially stable with*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|Y(t)\|_{\mathcal{H}} \leq -\Big(\lambda - \frac{1}{4}\beta^2 - \frac{M^2}{2}\|B\|^2\Big) \qquad a.s.$$

Next let us proceed by considering the point delay and distributed delay in (3.1) separately.

(I) Let $A_2(\cdot) \equiv 0$ in (3.1), then we have for any $\beta \in \mathbb{R}$ that $(\mathcal{A} + \beta\mathcal{B})\phi = ((A + \beta B)\phi_0 + A_1\phi_1(-r), d\phi_1(\theta)/d\theta)$. Hence, by virtue of Propositions 2.1 and 3.1, we have that if there exist numbers $\alpha > 0$, $\beta \in \mathbb{R}$ and $M > 0$ such that for some $\lambda \in (0, \alpha)$,

$$\begin{gathered} \|e^{t(A+\beta B)}\| \leq Me^{-\alpha t}, \qquad \beta^2 + 2M^2\|B\|^2 \leq 4\lambda, \\ \Big[\frac{\|A_1\|^2 e^{2\lambda r}}{(1-\|G\|)(1-e^{2\lambda r}(\|G\|+\|G\|^2))}\Big]^{1/2} < \alpha - \lambda, \end{gathered} \tag{3.3}$$

then the solution Y of (3.2) is exponentially stable in the almost sure sense. In this case, the delay parameter $r > 0$ satisfies

$$r < \frac{1}{2\lambda} \ln \frac{(\alpha-\lambda)^2(1-\|G\|)}{\|A_1\|^2 + (\alpha-\lambda)^2(1-\|G\|)(\|G\|+\|G\|^2)}, \tag{3.4}$$

whenever $\|G\| \neq 0$ or $\|G\| = 0$ and $\|A_1\| \neq 0$.

Example 3.1 (*Example in the End of Section 1 Revisited*)**.** First consider the following stochastic retarded partial differential equation,

$$\begin{cases} dy(t,\xi) = \dfrac{\partial^2}{\partial\xi^2}y(t,\xi)dt + \alpha\dfrac{\partial^2}{\partial\xi^2}y(t-r,\xi)dt + \sigma y(t,\xi)dw(t), \quad t \geq 0, \quad \xi \in [0,\pi], \\ y(t,0) = y(t,\pi) = 0, \ t \geq 0, \ \ y(0,\cdot) = \phi_0(\cdot) \in L^2(0,\pi), \ \ y_0(\cdot,\cdot) = \phi_1(\cdot,\cdot) \in L^2([-r,0]; L^2(0,\pi)), \end{cases} \tag{3.5}$$

where α, $\sigma \in \mathbb{R}$. It is known that if $r = 0$ and $\alpha - 1 < \sigma^2/2$, the trivial solution of (3.5) is exponentially stable in the almost sure sense. If $r > 0$, it is true that, in addition to the condition $\alpha - 1 < \sigma^2/2$, the pathwise exponential stability of the trivial solution of (3.5) is not sensitive to small delays $r > 0$ (see [4]). Next we want to consider a neutral type of version of (3.5), i.e., (1.5), which could be reformulated as

$$y(t) - \gamma y(t-r) = e^{tA}\phi_0 + (\alpha+\gamma)\int_0^t y(s-r)ds + \sigma\int_0^t y(s)dw(s),\ t \geq 0, \tag{3.6}$$

and $y_0 = \phi_1$, $\phi = (\phi_0, \phi_1) \in \mathcal{H} = H \times L^2([-r, 0], V)$, where $\gamma \in \mathbb{R}$, $V = H_0^1(0,\pi) \cap H^2(0,\pi)$, $H = L^2(0,\pi)$, $A = \Delta = \partial^2/\partial\xi^2$ and $\mathcal{B} \in \mathscr{L}(\mathcal{H})$ is given by $\mathcal{B}\phi = (\sigma\phi_0, 0)$ for $\phi = (\phi_0, \phi_1) \in \mathcal{H}$. Then $(\mathcal{A} + \beta\mathcal{B})\phi = (\Delta\phi_0 + \alpha\Delta\phi_1(-r) + \beta\sigma\phi_0, d\phi_1(\theta)/d\theta)$, $\phi \in \mathscr{D}(\mathcal{A})$. Suppose that $|\gamma| + |\gamma|^2 < 1$, then by virtue of (3.3), we have that if there exists some number $\beta \in \mathbb{R}$ such that for some $\lambda > 0$,

$$\lambda < 1 - \beta\sigma, \qquad \beta^2 + 2\sigma^2 \leq 4\lambda, \qquad \frac{(\alpha+\gamma)^2 e^{2\lambda r}}{(1-|\gamma|)(1-e^{2\lambda r}(|\gamma| + |\gamma|^2))} < (1-\beta\sigma-\lambda)^2, \tag{3.7}$$

then the trivial solution of (3.2) is exponentially stable in the almost sure sense. It may be verified that $\beta = -2\sigma$ and $\lambda = \frac{3}{2}\sigma^2 + \varepsilon$ with $\varepsilon > 0$ sufficiently small satisfy condition (3.7). In this case, the third inequality in (3.7) is actually reduced to

$$r < \frac{1}{3\sigma^2}\ln\frac{(1+\frac{\sigma^2}{2})^2(1-|\gamma|)}{(\alpha+\gamma)^2 + (1+\frac{\sigma^2}{2})^2(1-|\gamma|)(|\gamma|+|\gamma|^2)}. \tag{3.8}$$

For instance, if $\alpha = -\gamma$, $\sigma \neq 0$, the pathwise exponential stability of the lift-up equation of (3.6) is not sensitive to small delays $0 < r < -\frac{1}{3\sigma^2}\ln(|\gamma| + |\gamma|^2)$.

(II) Let us consider a stochastic system with distributed delay. To this end, we first state a proposition whose proof is analogous to that of Proposition 2.1.

Proposition 3.2. *Suppose that $\langle x, Ax\rangle_{V,V^*} \leq -\alpha\|x\|_V^2$ for all $x \in V$ and some $\alpha > 0$, and η takes the form $\eta(\tau) = -\int_\tau^0 A_2(\theta)d\theta$ with $A_2 \in L^2([-r,0], \mathscr{L}(V, V^*))$. Further, if $\|G\| + \|G\|^2 < 1$ and*

$$\Big[\frac{1}{(1-\|G\|)(1-e^{2\lambda r}(\|G\| + \|G\|^2))}\int_{-r}^0 e^{-2\lambda\theta}\|A_2(\theta)\|^2 d\theta\Big]^{1/2} < \alpha - \lambda \quad \text{for some} \quad 0 < \lambda < \alpha, \tag{3.9}$$

then there exists a constant $M > 0$ such that $\|e^{t\mathcal{A}}\| \leq Me^{-\lambda t}$ for all $t \geq 0$.

Example 3.2. Consider a distributed time delay version of (3.6) of the form: for $t \geq 0$, $\xi \in [0, \pi]$,

$$\begin{cases} d(y(t,\xi) - \gamma y(t-r,\xi)) = \Delta(y(t,\xi) - \gamma y(t-r,\xi))dt + \alpha\displaystyle\int_{-r}^0 y(t+\theta,\xi)d\theta dt + \sigma y(t,\xi)dw(t), \\ y(t,0) = y(t,\pi) = 0,\ t \geq 0,\ \ y(0,\cdot) = \phi_0(\cdot) \in L^2(0,\pi),\ y_0(\cdot,\cdot) = \phi_1(\cdot,\cdot) \in L^2([-r,0]; L^2(0,\pi)), \end{cases} \tag{3.10}$$

where $\Delta = \partial^2/\partial\xi^2$ and α, $\sigma \in \mathbb{R}$, which can be reformulated as

$$\begin{cases} y(t) - \gamma y(t-r) = e^{t\Delta}\phi_0 + \alpha\displaystyle\int_0^t\int_{-r}^0 y(s+\theta)d\theta ds + \sigma\int_0^t y(s)dw(s),\ \ t \geq 0, \\ y_0 = \phi_1,\ \phi = (\phi_0, \phi_1) \in \mathcal{H} = H \times L^2([-r,0], V). \end{cases} \tag{3.11}$$

Suppose that $|\gamma| + |\gamma|^2 < 1$, then we have similarly that if there is a value $\beta \in \mathbb{R}$ such that for some $\lambda > 0$,

$$\lambda < 1 - \beta\sigma, \qquad \beta^2 + 2\sigma^2 \leq 4\lambda, \qquad \frac{\alpha^2(e^{2r(1-\beta\sigma)} - 1)}{2(1-\beta\sigma)(1-|\gamma|)(1-e^{2\lambda r}(|\gamma|+|\gamma|^2))} < (1-\beta\sigma-\lambda)^2, \tag{3.12}$$

then the trivial solution of the lift-up equation of (3.11) is exponentially stable in the almost sure sense. It may be verified that if $\sigma \neq 0$, $\beta = -2\sigma$ and $\lambda = 1 + 2\sigma^2 - \varepsilon$ with $\varepsilon > 0$ sufficiently small satisfy condition (3.12). In this case, the third inequality in (3.12) is actually reduced to

$$r < \frac{1}{2 + 4\sigma^2} \ln \frac{\alpha^2 + 2(1 + \frac{\sigma^2}{2})^2(1 + 2\sigma^2)(1 - |\gamma|)}{\alpha^2 + 2(1 + \frac{\sigma^2}{2})^2(1 + 2\sigma^2)(1 - |\gamma|)(|\gamma| + |\gamma|^2)}. \tag{3.13}$$

That is, the pathwise exponential stability of the lift-up equation of (3.11) is not sensitive to small delays.

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